[6]Balas, M., "Discrete Time Stability of Continuous Time Controller Designs for Large Space Structures," *Journal of Guidance and Control,* Vol. 5, Sept.-Oct. 1982, pp. 541-543.
[7]Miller, R.K. and Michel, A.N., "Asymptotic Stability of Systems: Results Involving the System Topology," *SIAM Journal on Control and Optimization,* 18, 1980, pp. 181-190.

Modal Decoupling Conditions for Distributed Control of Flexible Structures

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Introduction

THE problem of the control of distributed parameter systems can be roughly divided into two approaches: 1) discretize the system in space and then use finite dimensional control theory; and 2) deal with the distributed model directly without discretizing. Recently, Meirovitch and Baruh[1] proposed a scheme for the optimal control of a certain class of conservative distributed parameter systems without resorting to discretization. In particular, they treated the control of self-adjoint conservative systems having known eigensolutions. It is the intent of this Note to point out that their results are applicable to a more general class of problems that includes nonconservative forces and to note that the necessary conditions are available for the existence of decoupling control laws. Decoupling control laws are defined to be those control laws dependent only on the modal state vector of the decoupled equation. This yields an infinite set of independent equations including the feedback control.

The use of decoupled controls allows the distributed parameter control problem to be solved by the independent modal-space control method.[1] This method allows each mode to be designed independently of other modes. As a result, the standard control problems involving optimal control and the regulator problem can be solved without difficulty. This method of control is not discussed in detail here, but is mentioned to supply motivation and application for the results that follow.

Class of Systems Considered

The class of flexible structures under consideration are those that may be successfully modeled by partial differential equations of the form

$$u_{tt}(x,t)+L_1u_t(x,t)+L_2u(x,t)=f(x,t) \text{ on } \Omega \tag{1}$$

subject to boundary conditions of the form $Bu(x,t)=0$ on $\partial\Omega$ and the usual initial conditions. Here, $u(x,t)$ represents the displacement of the point x in the bounded open region Ω in R^n, $n=1,2,3$, at time $t>0$. The region Ω is bounded by the boundary $\partial\Omega$ and the operator B is a linear spatial differential operator expressing the usual boundary conditions. The subscript t indicates partial differentiation with respect to time and L_1 and L_2 are linear partial differential operators in spatial coordinates. In order to insure the existence of a series converging to the solution of Eq. (1), several assumptions on the time invariant operators L_1 and L_2 must be made. Fortunately these assumptions are not too harsh and Eq. (1), along with the various assumptions on the operators L_1 and L_2, adequately describes the linear vibration of strings, beams, membranes, plates, etc.

Let $L_2(\Omega)$ denote the Hilbert space of all real-valued squared integrable functions on the domain Ω in the Lebesgue sense with the usual inner product and norm defined by

$$\langle u,v\rangle=\int_\Omega u(x)v(x)\,\mathrm{d}\Omega$$

and

$$\|u(x)\|=\langle u,u\rangle^{1/2}$$

respectively. Let L_1 and L_2 be time-invariant partial differential operators of order n_1 and n_2, respectively. Let $D(L)$ be the set of all functions $u(x,t)$ such that $Bu=0$ on $\partial\Omega$ and $u(x,t)$ and all of its derivatives up to the order $K=\max(n_1,n_2)$ are in $L_2(\Omega)$. The assumptions required for $u(x,t)$ to be expressed in terms of a convergent series in orthonormal eigenfunctions may now be concisely stated as follows.[2] If L_1 and L_2 are self-adjoint on $D(L)$ such that $L_1L_2=L_2L_1$ for all functions in $D(L)$ and if each operator has a compact resolvent,[3] then the solution of Eq. (1) may be written as the uniformly convergent series

$$u(x,t)=\sum_{n=1}^{\infty}a_n(t)\phi_n(x) \tag{2}$$

where the set $\{\phi_n(x)\}_{n=1}^{\infty}$ are the orthonormal eigenfunctions ($\langle\phi_n,\phi_m\rangle=\delta_{mn}$, the Kronecker delta) of the operator L_2 that are identical to the eigenfunctions of L_1.[4] The temporal functions $a_n(t)$ are assumed to be at least twice differentiable.

The restriction of requiring the coefficient operators to have compact resolvents will be satisfied if the stiffness operator (L_2) of the vibration problem of interest has an inverse defined by a Green's function. The requirement that the operators L_1 and L_2 commute is tantamount to restricting the class of problems considered to the class that can be solved by separation of variables.

The independent modal control method focuses on controlling the temporal functions $a_n(t)$. The work presented here concerns itself with the nature of the function $f(x,t)$ in Eq. (1) considered as a distributed control and how to choose $f(x,t)$ in such a way as to allow the method of independent modal space control to be used.

Previous Theory

In Ref. 1, systems given by Eq. (1) with $L_1=0$ are discussed and an optimal control method is developed for proportional control. This method is based on substitution of Eq. (2) into Eq. (1), multiplying by $\phi_m(x)$ and integrating over the domain Ω. This yields an infinite number of decoupled second-order ordinary differential equations of motion given by

$$\ddot{a}_n(t)+\lambda_n^2 a_n(t)=f_n(t) \tag{3a}$$

where λ_n^2 are the eigenvalues of L_2 associated with the eigenfunctions $\phi_n(x)$ and $f_n(t)$ is a generalized control force given by

$$f_n(t)=\int_\Omega \phi_n(x)f(x,t)\,\mathrm{d}\Omega \tag{3b}$$

Submitted April 25, 1983; revision received Aug. 3, 1983.

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Systems leading to independent equations such as (3) are said to be *internally decoupled* and form the basis for the work of Ref. 1. In addition, if $f_n(t)$ depends only on the nth mode $a_n(t)$, then Eqs. (3) are said to be *externally decoupled.* In general, $f_n(t)$ may depend on all of the coefficients $a_n(t)$, $n=1,2,3,\ldots$, due to feedback controls. However, in the event that Eqs. (3) are both internally and externally decoupled, a control system can be designed quite easily by the method of independent modal-space control. This process eliminates spillover when distributed controls are used and allows some rationale for minimum spillover with discrete controls.

Main Result

The purpose of this Note is to illustrate that the methods of Ref. 1 apply to a broader class of problems by deriving a necessary condition for externally decoupled controls and by pointing out the necessary and sufficient conditions for a given nonconservative structure to be internally decoupled.

The necessary and sufficient condition for a given system with damping ($L_1 \neq 0$) to be internally decoupled was given in Ref. 4 and is simply that $L_1L_2=L_2L_1$ on $D(L)$. Under this assumption, substitution of Eq. (2) into Eq. (1) yields

$$\ddot{a}_n(t)+\lambda_n^{(1)}\dot{a}_n(t)+\lambda_n^{(2)}a_n(t)=f_n(t), \quad n=1,2\ldots \tag{4}$$

where $\lambda_n^{(1)}$ are the eigenfunctions of the operator L_1. Note that this assumption is satisfied with proportional or Rayleigh damping.

In Ref. 1, independent modal-space control is illustrated for the case where the control forces are proportional to the state or mode to be controlled. Symbolically, it can be stated that Eq. (1) will be externally decoupled if

$$f_n(t)=\alpha_n a_n(t)+\beta_n\dot{a}_n(t)$$

where α_n and β_n are constants. However, it is a fairly simple matter to extend this and show that Eq. (1) will be both externally and internally decoupled if

$$f(x,t)=L_3u_t(x,t)+L_4u(x,t) \tag{5}$$

where L_3 and L_4 are self-adjoint, spatial differential operators with compact resolvents on $D(L)$ such that

$$L_iL_j=L_jL_i \qquad i,j=1,2,3,4 \tag{6}$$

for all $u(x,t)$ in the set $D(L)$.

To see that this result is true, note that if each L_k is self-adjoint with compact resolvent, then each L_k has a complete set of orthonormal eigenfunctions $\{\phi_n^k(x)\}_{n=1}^{\infty}$ and a discrete spectrum $\{\lambda_n^k\}_n^{\infty}=1$. Thus

$$L_k\phi_n^k=\lambda_n^k\phi_n^k, \qquad n=1,2,3,\ldots \tag{7}$$

Operating on this equality with L_j yields [ϕ_n is in $D(L)$]

$$L_j(\lambda_n^k\phi_n^k) \tag{8}$$

Since L_k and L_j commute, this becomes

$$L_k(L_j\phi_n^k)=\lambda_n^k(L_j\phi_n^k) \tag{9}$$

Equation (9) states that the function $L_j\phi_n^k$, which is in $D(L)$, is an eigenfunction of L_k having eigenvalue λ_n^k as its eigenvalue. Since the set $\{\phi_n^k\}_{n=1}^{\infty}$ is orthonormal, eigenfunctions belonging to the same eigenvalue must be proportional. Thus

$$L_j\phi_n^k=\gamma_n\phi_n^k \tag{10}$$

for all n and γ_n becomes the eigenvalue of L_j associated with the eigenfunction $\phi_n^k(x)$, i.e., $\lambda_n^k=\gamma_n$. Hence, commuting operators have the same eigenfunctions, i.e., $\phi_n^k=\phi_n^j$, and this set of eigenfunctions can be denoted simply $\{\phi_n\}_{n=1}^{\infty}$.

Another interesting property of commuting operators is that linear combinations of these operators result in operators with the same properties (self-adjoint, etc.), which have eigenvalues that are the same linear combination of the eigenvalues of each operator. To see this, calculate the eigenvalues of aL_i+bL_j, where a and b are constants. The eigenvalue problem is

$$(aL_i+bL_j)\phi_n=aL_i\phi_n+bL_2\phi_n=(a\lambda_n^i+b\lambda_n^j)\phi_n \tag{11}$$

for all ϕ_n.

Next consider substitution of the forcing function [Eq. (5)] into Eq. (1). This yields

$$u_{tt}(x,t)+L_1u_t(x,t)+L_2u(x,t)=L_3u_t(x,t)+L_4u(x,t) \text{ on } \Omega$$

which can be rewritten as the homogeneous equation

$$u_{tt}+(L_1-L_3)u_t+(L_2-L_4)u=0 \text{ on } \Omega \tag{12}$$

Assuming a solution of the form of Eq. (2), multiplying by $\phi_m(x)$, integrating over Ω, and using Eq. (11) yields

$$\ddot{a}_n(t)+(\lambda_n^1-\lambda_n^3)\dot{a}_n(t)+(\lambda_n^2-\lambda_n^4)a_n(t)=0, \quad n=1,2,3\ldots \tag{13}$$

Thus, under the assumption of commuting operators, the control problem is reduced to an infinite number of decoupled ordinary differential equations and Eq. (1) is both internally and externally decoupled.

Examples

The longitudinal vibration of a clamped bar with internal viscous damping immersed in a fluid provides an example of a distributed system with a distributed control, if the viscous damping rate provided by the fluid (denoted by γ) is considered as the proportional control gain. This problem provides a simple example to indicate the validity of the above stated results as well as to illustrate their use.

The equation of free vibration is given by

$$u_{tt}(x,t)-2b\frac{\partial^2}{\partial x^2}u_t(x,t)-a\frac{\partial^2}{\partial x^2}u(x,t)=0 \text{ on } \Omega=(0,1) \tag{14}$$

with boundary conditions $u(x,t)=0$ on $\partial\Omega$. Here $u(x,t)$ is the displacement of the bar and the positive constants b and a reflect the relevant physical parameters. If external damping is provided as a control, Eq. (13) becomes

$$u_{tt}(x,t)-2b\frac{\partial^2}{\partial x^2}u_t(x,t)-a\frac{\partial^2}{\partial x^2}u(x,t)=-2\gamma u_t(x,t) \text{ on } \Omega \tag{15}$$

where $-2\gamma u_t(x,t)$ represents the control force and γ the proportional control gain. The operators of Eqs. (1) and (5) are readily identified as:

$$L_1=-2b\frac{\partial^2}{\partial x^2},\ L_2=-a\frac{\partial^2}{\partial x^2},\ L_3=-2\gamma I, \text{ and } L_4=0$$

where I is the identity operator and each operator is defined on the domain $D(L)=\{u(x,t)$ in $L^2(0,1)$ [such that u_x and u_{xx} are continuous on (0,1) and $u(0)=u(1)=0.$]} Each operator is known to be self-adjoint and to have a Green's function inverse; thus, each has a compact resolvent. Furthermore, it can easily be seen that $L_iL_j=L_jL_i$ on $D(L)$ and that Eq. (13) becomes

$$\ddot{a}_n(t)+2(\gamma-bn^2\pi^2)\dot{a}_n(t)+an^2\pi^2a_n(t)=0 \tag{16}$$

where n is a positive integer. Equation (16) agrees with the results obtained by solving Eq. (15) using separation of variables.

As a second example, consider the transverse vibration of a simply supported bar that is given by Eq. (1) with $L_1=0$ and $L_2=a(\partial^4/\partial x^4)$. The domain $D(L)$ becomes the set of all real functions $u(x,t)$ such that u and its derivatives up through order four are continuous in $L^2(0,1)$ and satisfy the boundary conditions $u(0)=u(1)=u_{xx}(0)=u_{xx}(1)=0$. If a control force of the form

$$f(x,t)=-bu_t(x,t)+c\frac{\partial^2}{\partial x^2}u(x,t)$$

is used then $L_3=-bI$ and $L_4=c(\partial^2/\partial x^2)$. Again, each operator is self-adjoint and has a compact resolvent. A short calculation shows that Eq. (6) is satisfied and the parameters b and c can be chosen as control parameters to shape the response without using modal truncation.

Conclusion

Necessary conditions for a given nonconservative structure and feedback control system to be both externally and internally decoupled have been presented. These conditions allow the method of independent modal-space control to be applied to a larger class of structures.

Acknowledgment

This work was performed under Air Force Office of Scientific Research Grant AFOSR820242.

References

[1] Meirovitch, L. and Baruh, H., "Control of Self-Adjoint Distributed Parameter Systems," *Journal of Guidance, Control and Dynamics*, Vol. 5, Jan.-Feb. 1982, pp. 60-66.

[2] Inman, D. J. and Andry, A. N. Jr., "The Nature of the Temporal Solutions of Damped Distributed Systems with Classical Normal Modes," *Journal of Applied Mechanics*, Vol. 49, No. 4, pp. 867-870.

[3] Kato, T., *Perturbation Theory for Linear Operators*, Springer-Verlag, New York, 1966, p. 169.

[4] Caughey, T.K. and O'Kelly, M. E. J., "Classical Normal Modes in Damped Linear Systems," *Journal of Applied Mechanics*, Vol. 32, 1965, pp. 583-588.

Robust Nonlinear Least Squares Estimation Using the Chow-Yorke Homotopy Method

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Introduction

HOMOTOPY and continuation methods have recently found widespread adoption as approaches for developing exceptionally robust algorithms for many multidimensional problems in nonlinear solid and fluid mechanics and optimal control.[1-8] To a lesser extent, these ideas have been applied to nonlinear estimation and system identification problems; Refs. 9 and 10 are recent examples. It is important to remark that merely adopting a homotopy or a continuation approach guarantees neither efficiency nor robustness of the resulting algorithm. A marginal increase in the domain of numerical convergence in exchange for a large decrease in computational efficiency is a dubious justification for a homotopy or continuation method in comparison to, say, a Gauss-Newton method.

In the present paper, we demonstrate a recently developed homotopy algorithm,[3] and find an order of magnitude increase in the domain of convergence in comparison to the embedded Gauss-Newton algorithm of Kirszenblat and Chetrit.[10] Since Kirszenblat and Chetrit's algorithm was originally shown to be considerably more robust than the classical Gauss-Newton algorithm, we establish that the present Chow-Yorke algorithm is robust, indeed.

The Nonlinear Least Squares Problem

We consider the problem of determining the best estimate of a parameter vector $x=(x_1,x_2...x_n)^T$ which results in a given nonlinear function

$$y=f(t,x) \tag{1}$$

being a best fit of a given set of measured y-values

$$\{t_1,\tilde{y}_1;t_2,\tilde{y}_2;...;\ t_m,\tilde{y}_m\}, \qquad m>n \tag{2}$$

where the m measurement times t_j are assumed perfectly known. Adopting the simple least squares penalty function, we seek the estimate of x which minimizes

$$J(x)=\sum_{j=1}^{m}[\tilde{y}_j-f(t_j,x)]^2 \tag{3}$$

The classical Gauss-Newton algorithm[11] for obtaining successive corrections $\Delta x(i)$ to a sequence of trial vectors $x(i)$ is based upon taking $x=x(i)+\Delta x(i)$ and linearizing $f(t_j,x)$ about $x(i)$; upon substituting this linear approximation for $f(t_j,x)$ into Eq. (3), the resulting quadratic (in Δx) approximation of J can be minimized with respect to Δx to obtain the normal equations

$$\Delta x(i)=(A^TA)^{-1}A^T\Delta y; \qquad i=0,1,2,... \tag{4}$$

where

$$[A]=\begin{bmatrix} \left.\dfrac{\partial f(t_1,x)}{\partial x_1}\right|_{x(i)} & \cdots & \left.\dfrac{\partial f(t_1,x)}{\partial x_n}\right|_{x(i)} \\ \vdots & & \vdots \\ \left.\dfrac{\partial f(t_m,x)}{\partial x_1}\right|_{x(i)} & \cdots & \left.\dfrac{\partial f(t_m,x)}{\partial x_n}\right|_{x(i)} \end{bmatrix} \tag{5}$$

is the locally evaluated Jacobian matrix and

$$\Delta y=\{[\tilde{y}_1-f(t_1,x(i))],\ \ ...[\tilde{y}_m-f(t_m-f(t_m,x(i))]\}^T \tag{6}$$

is the residual vector.

The Continuation Method of Kirszenblat and Chetrit

Motivated by the desire to enlarge the domain of convergence of the Gauss-Newton algorithm [successive corrections using Eq. (4)], Kirszenblat and Chetrit[10] imbedded the Gauss-Newton algorithm into a continuation

Received July 6, 1983; revision received Aug. 29, 1983.

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